Lecture 33

Measure and Integration

I. K Rana

24/3/11

$$f \in L_1(X, \mathcal{S}, \mu)$$

$$\iff \mathrm{Re}(f),\ \mathrm{Im}(f) \in L_1^{\mathbb{R}}(X)$$

$$|f| = \sqrt{(\mathrm{Re}(f))^2 + (\mathrm{Im}(f))^2}$$

$$\leq \sqrt{2}\,\big(|\mathrm{Re}(f)| + |\mathrm{Im}(f)|\big)$$

(Ex)

$$\implies |f| \in L_1^{\mathbb{R}}(X)$$

Let $|f| \in L_1^{\mathbb{R}}(X)$

Then

$$|\mathrm{Re}(f)| \leq |f|$$

$$|\mathrm{Im}(f)| \leq |f|$$

$$\Longrightarrow \quad \mathrm{Im}(f), \mathrm{Re}(f) \in L_1^{\mathbb{R}}(X)$$

$$\Longrightarrow \quad f \in L_1(X, \mathcal{S}, \mu)$$

Hence $\quad f \in L_1(X) \Longleftrightarrow |f| \in L_1^{\mathbb{R}}(X)$

<u>Claim</u>

$$\left|\int f\,d\mu\right| \leq \int |f|\,d\mu \,?$$

Let $\alpha := \left|\int f\,d\mu\right|$

Let $\int f\,d\mu = \alpha e^{i\theta}, \; 0 \leq \theta < 2\pi$

$$\Longrightarrow \quad \alpha = e^{-i\theta} \int f\,d\mu$$

$$= \int (e^{-i\theta} f)\,d\mu \quad (!)$$

Let $e^{-i\theta} f := f_1 + i f_2$

Then

$$\alpha = \int (e^{-i\theta} f)\, d\mu$$

$$= \int f_1\, d\mu + i \int f_2\, d\mu$$

$$\Rightarrow \int f_2\, d\mu = 0 \text{ and}$$

$$\alpha = \int f_1\, d\mu = \left| \int f_1\, d\mu \right|$$

$$\leq \int |f_1|\, d\mu$$

$$f_1 = \mathrm{Re}(e^{-i\alpha} f)$$

$$|f_1| \leq |e^{-i\alpha} f|$$

$$= |f|$$

$$\Rightarrow \int |f_1| \, d\mu \leq \int |f| \, d\mu$$

$$\Rightarrow \left| \int f \, d\mu \right| \leq \int |f| \, d\mu. \qquad \blacksquare$$

$f \in L_1(X)$, $\alpha \in \mathbb{C}$, then

$$\alpha f \in L_1(X)$$

$$\alpha = a + ib$$

$$\alpha f = (a+ib)\left(\mathrm{Re}(f) + i\,\mathrm{Im}(f)\right)$$

$$= \left(a\,\mathrm{Re}(f) - b\,\mathrm{Im}(f)\right)$$

$$+ i\left(b\,\mathrm{Re}(f) + a\,\mathrm{Im}(f)\right)$$

$$f \in L_1(X) \Longrightarrow \mathrm{Re}(f), \mathrm{Im}(f) \in L_1^{\mathbb{R}}$$

$$\Longrightarrow \alpha f \in L_1^{\mathbb{C}}(X)$$

$$\int (\alpha f)\, d\mu = \int \big(a\, \mathrm{Re}(f) - b\, \mathrm{Im}(f)\big)\, d\mu$$

$$+\, i \int \big(b\, \mathrm{Re}(f) + a\, \mathrm{Im}(f)\big)\, d\mu$$

$$= a \int \mathrm{Re}(f)\, d\mu - b \int \mathrm{Im}(f)\, d\mu$$

$$+\, i\, b \int \mathrm{Re}(f)\, d\mu + i\, a \int \mathrm{Im}(f)\, d\mu$$

$$= (a + i b) \left(\int \mathrm{Re}(f)\, d\mu + i \int \mathrm{Im}(f)\, d\mu \right)$$

$$= \alpha \left(\int f\, d\mu \right)$$

$$f, g \in L_1(X) \Rightarrow f+g \in L_1(X)$$

<u>**Ntg**</u>

$$|f+g| \leq |f| + |g|$$

$$\Rightarrow \int |f+g| \, d\mu \leq \underbrace{\int |f| \, d\mu + \int |g| \, d\mu}_{< +\infty}$$

$$\Rightarrow \quad f+g \in L_1(X).$$

$$\mathrm{Re}(f+g) = \mathrm{Re}(f) + \mathrm{Re}(g)$$

$$\mathrm{Im}(f+g) = \mathrm{Im}(f) + \mathrm{Im}(g)$$

$$\begin{aligned}
\int (f+g)\,d\mu &= \int \mathrm{Re}(f+g)\,d\mu \\
&\quad + \int \mathrm{Im}(f+g)\,d\mu \\
&= \int \big(\mathrm{Re}(f) + \mathrm{Re}(g)\big)\,d\mu \\
&\quad + i\int \big(\mathrm{Im}(f) + \mathrm{Im}(g)\big)\,d\mu \\
&= \int \mathrm{Re}(f)\,d\mu + \int \mathrm{Re}(g)\,d\mu \\
&\quad + i\int \mathrm{Im}(f)\,d\mu + i\int \mathrm{Im}(g)\,d\mu \\
&= \int f\,d\mu + \int g\,d\mu.
\end{aligned}$$

$f \in L_1(X)$, $E \in \mathcal{S}$

$$\chi_E f = \chi_E\big(\mathrm{Re}(f) + i\,\mathrm{Im}(f)\big)$$

$$= \underbrace{\chi_E \mathrm{Re}(f)}_{} + i\,\underbrace{\chi_E \mathrm{Im}(f)}_{}$$

$$\int \chi_E f := \int_E f\,d\mu = \int \chi_E \mathrm{Re}(f)\,d\mu$$

$$+\, i \int \chi_E \mathrm{Im}(f))$$

$$= \int_E \mathrm{Re}(f) + i \int_E \mathrm{Im}(f)\,d\mu$$

$f \in L_1(X)$, $E \in \mathcal{S}$

$$\int_E f\,d\mu = \int_E \mathrm{Re}(f)\,d\mu + i\int_E \mathrm{Im}(f)\,d\mu$$

Suppose $E_1, E_2 \in \mathcal{S}$; $E_1 \cap E_2 = \varphi$.

$$\int_{E_1 \sqcup E_2} f\,d\mu = \int_{E_1 \sqcup E_2} \mathrm{Re}(f)\,d\mu + i\int_{E_1 \sqcup E_2} \mathrm{Im}(f)\,d\mu$$

$$= \int_{E_1} \mathrm{Re}(f)\,d\mu + \int_{E_2} \mathrm{Re}(f)\,d\mu$$

$$+ i \int_{E_1} \mathrm{Im}(f)\,d\mu + i \int_{E_2} \mathrm{Im}(f)\,d\mu$$

$$= \int_{E_1} f\,d\mu + \int_{E_2} f\,d\mu \, .$$

$$E = \bigsqcup_{i=1}^{\infty} E_i\,, \quad E_i \in \Sigma.$$

$$f \in L_1(X). \iff |f| \in L_1^{\wedge}$$

$\displaystyle\sum_{i=1}^{\infty} \int_{E_i} f\,d\mu.$ is absolutely convergent:

$\displaystyle\sum_{i=1}^{\infty} \left| \int_{E_i} f\,d\mu \right|$ is cgt ?

Enough to show $\displaystyle\sum_{i=1}^{\infty} \int_{E_i} |f|\,d\mu < +\infty$ ✓

__Claim__

$$\int_E f\,d\mu = \sum_{i=1}^{\infty} \int_{E_i} f\,d\mu \ ?$$

$$\left| \int_E f\,d\mu - \sum_{i=1}^{n} \int_{E_i} f\,d\mu \right|$$

$$\cancel{= \left| \sum_{i=1}^{n} \int_E f\,d\mu - \int_{E_i} f\,d\mu \right|}$$

$$= \left| \int_E f\,d\mu - \int_{\bigsqcup_{i=1}^n E_i} f\,d\mu \right|$$

$$= \left| \int_E f\,d\mu - \int \chi_{\bigsqcup_{i=1}^n E_i} f\,d\mu \right|$$

$$\le \int \underbrace{\left|\chi_E f - \chi_{\bigsqcup_{i=1}^n E_i} f\right|}\,d\mu.$$

$$\chi_{\bigsqcup_{i=1}^{n} E_i} f \longrightarrow \chi_E f$$

$$\left|\chi_E f - \chi_{\bigsqcup_{i=1}^{n} E_i} f\right| \longrightarrow 0 \quad \text{as } n \to \infty$$

and $\left|\chi_E f - \chi_{\bigsqcup_{i=1}^{n} E_i} f\right| \le 2|f|$

$|f| \in L_1^{\mu}$.

By D C Thm

$$\left| \int_E f \, d\mu - \sum_{i=1}^{n} \int_{E_i} f \, d\mu \right| \longrightarrow 0$$

as $n \longrightarrow \infty$

Hence

$$\int_E f \, d\mu = \sum_{i=1}^{\infty} \int_{E_i} f \, d\mu$$

$\square$

Pf.

$$|f_n| \leqslant g \in L_1^{\Sigma}$$

$$\Rightarrow f_n \in L_1(X)$$

$$f_n \longrightarrow f \text{ a.e}$$

$$\Rightarrow |f| \leq g \text{ a.e}$$

$$\Rightarrow f \in L_1(X).$$

$$|\mathrm{Re}(f_n)| \leq |f_n| \leq g$$

<u>DCThm</u>

$$\int \mathrm{Re}(f_n)\,d\mu \longrightarrow \int \mathrm{Re}(f)\,d\mu$$

$$\parallel^{\mu} \quad i\int \mathrm{Im}(f_n)\,d\mu \longrightarrow i\int \mathrm{Im}(f)\,d\mu$$

$$\Longrightarrow \int f_n\,d\mu \longrightarrow \int f\,d\mu.$$

$\square$